9-29-21

Last time: limits

A function f has $\lim_{\bar{x}\to\bar{a}} f(\bar{x}) = L$ iff. for all cont. space curves, $\vec{r}(t)$ with $\lim_{t\to 0} \vec{r}(t) = \bar{a}$.

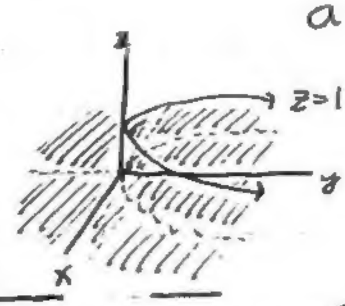
- limit from direction of curve L

*To show limit is DNE, find $\vec{r}_0(t)$ and $\bar{r}_1(t)$ with $\lim_{t\to 0^+} \bar{r}_2(t) = \vec{a}$ and show $\lim_{t\to 0^+} f(\bar{r}_0(t)) \neq \lim_{t\to 0^+} f(r_1(t))$

Use lines $\ell_{a,b}(t) = \vec{a} + t\langle a,b\rangle$

These lines DONT suffice showing ~~~~ if a limit ~~~~ exists.

Let $f(x,y) = \begin{cases} 1 & \text{if } y = x^2 \\ 0 & \text{otherwise} \end{cases}$



Limiting + $\bar{a} = \bar{0}$ along the line $\ell_{a,b}(t)$, notice $f(\ell_{a,b}(t)) = f(at, bt) = 0$ for all $t > 0$ ~~~~

~~~~ $(at)^2 = bt$ has at most 2 solutions

$$\lim_{t\to 0^+} f(\ell_{a,b}(t)) = \lim_{t\to 0^+} 0 = 0$$

$\bar{r}(t) = \langle t, t^2\rangle$, we see: $f(\bar{r}(t)) = f(t, t^2) = 1$ for all t.

$$\lim_{t\to 0^+} f(\bar{r}(t)) = \lim_{t\to 0^+} 1 = 1$$

*As $1 \neq 0$, $\lim_{\bar{x}\to\bar{0}} f(\bar{x})$ DNE by curves criterion ▨

Q: How do w show a limit exists?

TRICK: Use Polar Coordinates (doesn't always work)
~~~~

9-29-21

Limits cont...

<u>Ex:</u> Does $\lim_{\vec{x}\to\vec{0}} \frac{\sin(x^2+y^2)}{x^2+y^2}$ exist? (yes, but how)

1. Convert to Polar Coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$(x,y)\to(0,0)$ iff. $r\to0^+$

Polar coordinates use pos. radius, so we approach 0^+.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to0^+} \frac{\sin((r\cos\theta)^2+(r\sin\theta)^2)}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to0^+} \frac{\sin(r^2(\cos^2\theta+\sin^2\theta))}{r^2(\cos^2\theta+\sin^2\theta)} = \lim_{r\to0^+} \frac{\sin(r^2)}{r^2}$$ * $\frac{0}{0}$ type

[$\cos^2\theta+\sin^2\theta=1$]

$\boxed{H}$

$$\lim_{r\to0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to0^+} \cos(r^2) = \cos(0^2) = \cos(0) = 1$$ ☒

<u>Ex:</u> Does $\lim_{\vec{x}\to\vec{0}} \frac{x^2-y^2}{x^2+y^2}$ exist? (Polar coordinate trick)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{r\to0^+} \frac{(r\cos\theta)^2-(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2} = \lim_{r\to0^+} \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2(\cos^2\theta+\sin^2\theta)}$$

$$\lim_{r\to0^+} \cos(2\theta) = \cos(2\theta)$$ *(depending on r, not θ)*

Approaching $\theta = \pi/2$, expect $\lim_{\vec{x}\to\vec{0}} f(x,y) = \cos(2\frac{\pi}{2}) = -1$

Approaching $\theta = 0$, expect $\lim_{\vec{x}\to\vec{0}} f(x,y) = \cos(0) = 1$

∴ Limit does not exist by Curves criterion ☒

9-29-21

Limits cont...

Continuity

- Function f is continuous @ $\vec{a} \in dom(f)$ when $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$
  - f is continuous @ set D when f is continuous @ every $\vec{a} \in D$

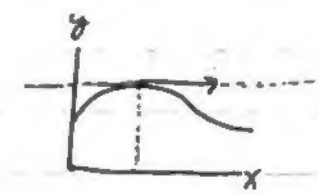
Ex: Every polynomial is cont. everywhere.
Ex: Every rational function is cont. on its domain.
Ex: $f(x,y)\ \frac{\sin(x^2+y^2)}{x^2+y^2}$ is cont. everywhere except $(0,0)$.

Cont. everywhere: $g(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$

NB: The "usual" rules for continuity from Calculus I still apply

Derivatives of Multivariable Functions

Idea: Derivative measures how function changes with small changes in input. *In a given direction.*



Defn: Directional derivative of function f of n variables @ $\vec{a} \in dom(f)$ in the direction of unit vector $\vec{u} \in \mathbb{R}^2$ is:

$$D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$$

Ex: Compute $D_{\bar{u}} f(a)$ for $f(x,y) = x\sqrt{y}$ @ $\vec{a} = \langle 2,4 \rangle$ in direction $\bar{v} = \langle 2,-1 \rangle$.

$$\bar{u} = \frac{1}{|\bar{v}|}\bar{v} = \frac{1}{\sqrt{5}}\langle 2,-1 \rangle = \langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$$

y

9-29-21

Limits cont...

Ex cont...

$$D_{\hat{u}} f(a) = \lim_{h \to 0^+} \frac{f(2+\frac{2}{\sqrt{5}}h, 4-\frac{1}{\sqrt{5}}h) - f(2,4)}{h}$$

$$= \lim_{h \to 0^+} \frac{(2+2/\sqrt{5}h)\sqrt{4-\frac{1}{\sqrt{5}}h} - 2\cdot\sqrt{4}}{h}$$

$$= \lim_{h \to 0^+} \frac{(2+\frac{2}{\sqrt{5}}h)\sqrt{4-\frac{1}{\sqrt{5}}h} - 4}{h} \cdot \frac{-(2+\frac{2}{\sqrt{5}}h)\sqrt{4-\frac{1}{\sqrt{5}}h} - 4}{-(2+\frac{2}{\sqrt{5}}h)\sqrt{4-\frac{1}{\sqrt{5}}h} - 4}$$

$$= \lim_{h \to 0^+} \frac{-(2+\frac{2}{\sqrt{5}}h)^2(4-\frac{1}{\sqrt{5}}h) + 16}{-h\underbrace{(4+(2+\frac{2}{\sqrt{5}}h)\sqrt{4-\frac{1}{\sqrt{5}}h})}_{=\boxtimes}} = \lim_{h \to 0^+} \frac{-(4 \cdot \frac{8}{\sqrt{5}}h \cdot \frac{4}{5}h^2)(4-\frac{1}{\sqrt{5}}h) + 16}{\boxtimes}$$

$$= \lim_{h \to 0^+} \frac{-(16 - \frac{4}{\sqrt{5}}h + \frac{32}{\sqrt{5}}h - \frac{8}{5}h^2 + \frac{16}{5}h^2 - \frac{4}{5\sqrt{5}}h^3) + 16}{\boxtimes}$$

$$= \lim_{h \to 0^+} \frac{\cancel{h}(-\frac{28}{\sqrt{5}} - (8/5)h + \frac{4}{5\sqrt{5}}h^2)}{-\cancel{h}(4+(2+\frac{2}{\sqrt{5}}h)\sqrt{4-1/\sqrt{5}h})} = \frac{-\frac{28}{\sqrt{5}} - 0 + 0^2}{-8\sqrt{5}} = 7/2\sqrt{5} \quad \boxtimes$$